Brane related relativistic Chaplygin gas with field-dependent Poincaré symmetry

M. Hassaïne and P. A. Horváthy
Laboratoire de Mathématiques et de Physique Théorique
Université de Tours, Parc de Grandmont
F-37200 TOURS (France)

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Abstract

A relativistic generalization of the Chaplygin gas is derived in a Kaluza-Klein framework, using a quadratic Lagrangian. Our theory admits the field-dependent Poincaré symmetry of the d-brane related Born-Infeld models of Bordemann and Hoppe, and of Jackiw and Polychronakos, to which it is secretely (but not manifestly) equivalent. Our action is in fact related to the usual Nambu-Goto action [world volume] of d-branes in the same way as the Polyakov and the Nambu action are in strings theory.

1 Introduction

In the light-cone gauge, a relativistic d-brane moving in $(d+1,1)$ dimensional Minkowski space yields a $(d,1)$ dimensional isentropic and irrotational fluid, called the Chaplygin gas [1, 2], whose manifest galilean symmetry extends into a non-linearly realized $(d+1,1)$-dimensional Poincaré dynamical symmetry [1, 2, 3].

Recently [2], Jackiw and Polychronakos presented a relativistic generalization of the Chaplygin gas, with the Lagrange density

$$L^{\mathrm{JP}} = \Theta\,\partial_\tau\rho - \sqrt{\rho^2c^2+a^2}\sqrt{c^2+(\vec{\nabla}\Theta)^2}, \tag{1.1}$$

where τ denotes the relativistic time, Θ is the momentum (!) potential, ρ the density, and the constant a is the interaction strength. This specific form is chosen so that the non-relativistic Chaplygin model is recovered in the limit $c\to\infty$. In what follows, we set $c=1$, and focus our attention to the relativistic model. The equations of motion associated to (1.1) read

$$\begin{cases} \partial_\tau \rho + \vec{\nabla} \cdot \left(\vec{\nabla}\Theta \sqrt{\dfrac{\rho^2 + a^2}{1 + (\vec{\nabla}\Theta)^2}} \right) = 0, \\ \partial_\tau \Theta + \rho \sqrt{\dfrac{1 + (\vec{\nabla}\Theta)^2}{\rho^2 + a^2}} = 0. \end{cases} \tag{1.2}$$

Evaluating ρ using the second equation allows us to present the Lagrange density (1.1) in the Born-Infeld form[1],

$$L^{\text{Born-Infeld}} = -a\sqrt{1 + \partial_\alpha \Theta \, \partial^\alpha \Theta}, \tag{1.3}$$

studied before also by Bordemann and Hoppe [4].

The manifest $(d, 1)$-dimensional Poincaré symmetry of (1.1) extends, just like for its non-relativistic counterpart, to a field-dependent $(d + 1, 1)$-dimensional Poincaré dynamical symmetry. The additional symmetries are time reparametrization, $\widetilde{x} = x$,

$$\begin{aligned} \widetilde{\tau} &= \frac{\tau}{\cosh \omega} + \Theta(\widetilde{\tau}, x) \tanh \omega, \\ \widetilde{\Theta} &= \frac{\Theta(\widetilde{\tau}, x)}{\cosh \omega} - \tau \tanh \omega, \end{aligned} \tag{1.4}$$

and space reparametrization, $\widetilde{\tau} = \tau$,

$$\begin{aligned} \widetilde{x} &= x - \hat{\gamma}\, \Theta(\tau, \widetilde{x}) \tanh \gamma + \hat{\gamma}(\hat{\gamma} \cdot x) \left(\frac{1 - \cos \gamma}{\cos \gamma} \right), \\ \tilde{\Theta} &= \frac{\Theta(\tau, \widetilde{x}) - (\hat{\gamma} \cdot x) \sin \gamma}{\cosh \gamma}, \end{aligned} \tag{1.5}$$

where $\gamma = |\vec{\gamma}|$ and $\hat{\gamma} = \vec{\gamma}/\gamma$.

In [5], we linearized the field-dependent Poincaré symmetry of the non-relativistic Chaplygin gas by unfolding the system into a "Kaluza-Klein" spacetime M, obtained by adding a coordinate s to non relativistic space and time, x and t. Then, identifying s with (minus) the transformed field $\widetilde{\Theta}$, all symmetries become Poincaré transformations of $(d+1, 1)$ dimensional Minkowski space M with metric $dx^2 + 2dtds$. (t and s are hence light-cone coordinates). Then we derived the non-relativistic Chaplygin model by *lightlike* reduction from a field theory, (3.1) below, defined on M viewed as a "Kaluza-Klein" spacetime [5].

Here we provide a similar interpretation of space- and time reparametrizations and derive, from the same theory, a relativistic model, using *spacelike reduction.* Remarkably, our quadratic action (3.1) is related to the Nambu-Goto action (4.3) of d-branes precisely as the Polyakov and the Nambu actions are for strings [8, 9]. The secret (but not manifest) equivalence with Jackiw and Polychronakos is established in the Discussion.

[1] Our metric is "mostly positive", $-d\tau^2 + dx^2$, the opposite of that in [2].

2 Unfolding

Let us start with the time reparametrizations, (1.4). Following the same recipe as in the non-relativistic case, let us add the new coordinate

$$\sigma = -\widetilde{\Theta} \qquad \Longrightarrow \qquad \widetilde{\sigma} = -\Theta. \tag{2.1}$$

Then (1.4) yields $\widetilde{x} = x$,

$$\begin{aligned} \widetilde{\tau} &= \cosh\omega\,\tau - \sinh\omega\,\sigma, \\ \widetilde{\sigma} &= \cosh\omega\,\sigma - \sinh\omega\,\tau. \end{aligned} \tag{2.2}$$

which is in fact a Lorentz transformation in the σ direction of Minkowski space with metric $-d\tau^2 + dx^2 + d\sigma^2$. ($\tau$ is hence timelike and σ spacelike). Switching to the light-cone coordinates $t = \frac{-\tau+\sigma}{2}$, $s = \frac{\tau+\sigma}{2}$, (2.2) becomes furthermore the non-relativistic time dilation $\widetilde{x} = x$, $\widetilde{t} = e^{\delta}t$, $\widetilde{s} = e^{-\delta}s$ [5].

Space reparametrizations admit a similar interpretation. Applying again our rule (2.1), (1.5) unfolds as a rotation $d+1$-dimensional space, $\tilde{\tau} = \tau$,

$$\begin{aligned} \widetilde{x} &= x - \hat{\gamma}\sin\gamma\,\sigma - \hat{\gamma}(\hat{\gamma}\cdot x)(1-\cos\gamma), \\ \tilde{\sigma} &= \cos\gamma\,\sigma - (\hat{\gamma}\cdot x)\sin\gamma. \end{aligned} \tag{2.3}$$

Interestingly, a $(d,1)$ dimensional Lorentz boost lifted to our extended space, $\tilde{\sigma} = \sigma$,

$$\begin{aligned} \widetilde{x} &= x + \hat{\beta}\sinh\beta\,\tau - \hat{\beta}(\hat{\beta}\cdot x)(1-\cosh\beta), \\ \tilde{\tau} &= \cosh\beta\,\tau + (\hat{\beta}\cdot x)\sinh\beta, \end{aligned} \tag{2.4}$$

($\beta = |\vec{\beta}|$, $\hat{\beta} = \vec{\beta}/\beta$) is related to the space reparametrization by the interchange of τ and σ and by changing γ into $i\beta$. (In the non-relativistic case, "antiboosts" and galilean boosts are related interchanging the light-cone coordinates s and t [5]).

3 Dynamics

In [5] we considered the fields ϱ and θ on the extended space M[2], described by the action

$$\int\Big(-\frac{1}{2}\varrho\,\partial_\mu\theta\,\partial^\mu\theta - V(\varrho)\Big)\,d^{d+2}x, \qquad V(\varrho) = \frac{\lambda}{\varrho}, \tag{3.1}$$

whose Euler-Lagrange equations read

$$\left\{\begin{array}{l} \partial_\mu(\varrho\,\partial^\mu\theta) = 0, \\ \frac{1}{2}\partial_\mu\theta\partial^\mu\theta = -V'(\varrho) = \dfrac{\lambda}{\varrho^2}. \end{array}\right. \tag{3.2}$$

[2]According to our conventions, i, j are spatial indices, $\alpha, \beta, \ldots$ refer to coordinates on ordinary spacetime, and $\mu, \nu, \ldots$ refer to the extended "Kaluza-Klein" spacetime, M.

Then the non-relativistic Chaplygin system can be derived by lightlike reduction from (3.1), when the projected fields Θ and ρ are defined by the conditions [5]

$$\begin{aligned} &\theta(x,t,-\Theta(x,t)) = 0, \\ &\rho(x,t) = \varrho(x,t,-\Theta(x,t))\partial_s\theta(x,t,-\Theta(x,t)). \end{aligned} \tag{3.3}$$

The manifest Poincaré symmetry of (3.1) survives the reduction and yields the field-dependent Poincaré dynamical symmetry of the non-relativistic model.

Let us now derive a new, relativistic model obtained from the same theory, but by spacelike reduction. Let us hence consider the relativistic coordinates x, τ, σ on Minkowski space. Then, replacing the rules (3.3) by their relativistic version, $t \to \tau$, $s \to \sigma$, the action (3.1) and the equations of motion (3.2) project, again for $V \propto 1/\varrho$ [only], to the manifestly $(d,1)$-dimensional Poincaré invariant expressions

$$L = \frac{1}{2}\rho\left((\partial_\tau\Theta)^2 - (\vec{\nabla}\Theta)^2 - 1\right) - \frac{\lambda}{\rho}, \tag{3.4}$$

$$\begin{cases} \partial_\tau(\rho\,\partial^\tau\Theta) + \vec{\nabla}\cdot(\rho\vec{\nabla}\Theta) = 0, \\ -(\partial_\tau\Theta)^2 + (\vec{\nabla}\Theta)^2 + 1 = \dfrac{2\lambda}{\rho^2}. \end{cases} \tag{3.5}$$

Although both our Lagrange density (3.4) and equations of motion (3.5) look rather different from those proposed by Jackiw and Polychronakos [2], (1.1) and (1.2), respectively, eliminating ρ using the lower equation in (3.5),

$$\rho = \frac{\sqrt{2\lambda}}{\sqrt{-(\partial_\tau\Theta)^2 + (\vec{\nabla}\Theta)^2 + 1}}, \tag{3.6}$$

allows us to recast our Lagrange density (3.4) in the same Born-Infeld form as in (1.3), $L = -\sqrt{2\lambda}\sqrt{-(\partial_\tau\Theta)^2 + (\vec{\nabla}\Theta)^2 + 1} \equiv -\sqrt{2\lambda}\sqrt{1 + \partial_\alpha\Theta\partial^\alpha\Theta}$.

Our reduced theory is still $(d+1,1)$-Poincaré invariant, as it can be shown along the same lines as in [5]. The conserved quantity associated to an infinitesimal Poincaré transformation (X^μ) of M is readily found using the [symmetric] energy-momentum tensor of (3.1) [5],

$$\begin{aligned} &Q = \int \frac{T^\tau{}_\mu X^\mu}{\partial_\sigma\theta}\, d^d x, \\ &T_{\mu\nu} = -\varrho\,\partial_\mu\theta\partial_\nu\theta + g_{\mu\nu}\left(\tfrac{1}{2}\rho\partial_\omega\theta\partial^\omega\theta + V(\varrho)\right). \end{aligned} \tag{3.7}$$

In detail, we get

$$
\begin{aligned}
&\mathcal{H} = \tfrac{1}{2}\rho[(\partial_\tau\Theta)^2 + (\vec{\nabla}\Theta)^2 + 1] + \frac{\lambda}{\rho}, && \text{energy}\\
&\mathcal{P}_i = -\rho\partial_i\Theta\,\partial_\tau\Theta, && \text{momentum}\\
&\mathcal{N} = -\rho\partial_\tau\Theta, && \text{relat. ``number''}\\
&\mathcal{D} = \mathcal{H}\Theta + \mathcal{N}\tau, && \text{time reparametrization}\\
&\mathcal{G}_i = x_i\mathcal{N} + \Theta\mathcal{P}_i && \text{space reparametrization}
\end{aligned}
\tag{3.8}
$$

For comparison, let us list those of (1.1),

$$
\begin{aligned}
&\mathcal{H}^{\mathrm{JP}} = \sqrt{(\rho^{\mathrm{JP}})^2 + a^2}\,\sqrt{\left(\vec{\nabla}\Theta^{\mathrm{JP}}\right)^2 + 1} && \text{BI energy}\\
&\mathcal{P}_i^{\mathrm{JP}} = \rho^{\mathrm{JP}}\,\partial_i\Theta^{\mathrm{JP}}, && \text{BI momentum}\\
&\mathcal{N}^{\mathrm{JP}} = \rho^{\mathrm{JP}}, && \text{BI number}\\
&\mathcal{D}^{\mathrm{JP}} = \mathcal{H}^{\mathrm{JP}}\Theta^{\mathrm{JP}} + \rho^{\mathrm{JP}}\tau, && \text{time reparametrization}\\
&\mathcal{G}_i^{\mathrm{JP}} = x_i\rho^{\mathrm{JP}} + \Theta^{\mathrm{JP}}\mathcal{P}_i^{\mathrm{JP}} && \text{space reparametrization}
\end{aligned}
\tag{3.9}
$$

The energy and the momentum are different, and, in our case, it is $\mathcal{N}$, associated to the vertical translation, which plays the role of the BI particle density ρ^{JP}. Let us also observe that their momentum density is our $T^\sigma_i/\partial_\sigma\theta = \rho\,\partial_i\Theta$. The Lorentz boosts and the angular momentum read, in both cases, $\mathcal{L}_i = \tau\mathcal{P}_i - x_i\mathcal{H}$ and $\mathcal{M}_{ij} = x_i\mathcal{P}_j - x_j\mathcal{P}_i$, respectively. (The same results can also be recovered using Noether's theorem directly).

Somewhat paradoxically, both *relativistic* systems, (1.1) and (3.4), are also Galilei-invariant, simply because the $(d,1)$ dimensional Galilei group is a subgroup of the Poincaré group in $(d+1,1)$ dimensions. Applying our rules backwards, for a galilean boost we get, e. g., the field-dependent action

$$
\begin{aligned}
\widetilde{x} &= x - \tfrac{1}{2}\alpha\tau - \tfrac{1}{2}\alpha\widetilde{\Theta},\\
\widetilde{\tau} &= (1 + \tfrac{1}{4}\alpha^2)\tau - \alpha\cdot x + \tfrac{1}{4}\alpha^2\widetilde{\Theta},\\
\Theta &= \widetilde{\Theta}(1 + \tfrac{1}{4}\alpha^2) + \alpha\cdot x - \tfrac{1}{4}\alpha^2\tau + \widetilde{\Theta}(1 + \tfrac{1}{4}\alpha^2).
\end{aligned}
\tag{3.10}
$$

4 Relation to d-branes

Our framework here is closely related to the so-called non-parametric representation of d branes [4]. Our "vertical" variable σ (*alias* the field $-\Theta$) is in fact the z coordinate of the d-brane propagating in $(d+1,1)$ dimensional Minkowski space, and our "lifted" field θ is (minus) their u, the function whose level sets describe the d-brane as $\theta(x,\tau,z(x,\tau)) = 0$, — which is our first condition in (3.3).

In terms of θ, the motion of the d-brane is governed by the action

$$\int \sqrt{\partial_\mu\theta\,\partial^\mu\theta}\,d^{d+2}x, \tag{4.1}$$

whose equations of motion read

$$\partial_\mu\left(\frac{\partial^\mu\theta}{\sqrt{\partial_\nu\theta\,\partial^\nu\theta}}\right) = 0. \tag{4.2}$$

The integrand here is in fact the "Nambu" world volume of the d-brane [6],

$$\sqrt{\partial_\mu\theta\,\partial^\mu\theta} = \sqrt{\det(G_{\alpha\beta})}, \qquad G_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu. \tag{4.3}$$

The point is that one can get rid of the square root, just like for a free relativistic particle. (This latter can be described either by the usual invariant length action $-m\int\sqrt{-\dot{x}^2}\,d\tau$, or by a quadratic action plus a constraint, when an auxiliary variable is added [9]). Let us hence enlarge our pure scalar theory involving θ alone by introducing an auxiliary field we call ρ, required to satisfy (3.6), viewed as a *constraint*. Then (4.2) and (3.6) together are readily seen to imply the first equation in (3.2); but both equations (3.2) derive from our quadratic Lagrangian (3.1). Conversely, inserting ρ into our action and equations of motion, (4.1) and (4.2) are recovered. (The two-dimensional analog is string theory, where the quadratic Polyakov action can be used instead of the Nambu-Goto expression [8, 9]).

Choosing instead the auxiliary constraint

$$\rho^{\mathrm{JP}} = \frac{a\,\partial_\tau\Theta}{\sqrt{-(\partial_\tau\Theta)^2 + (\vec{\nabla}\Theta)^2 + 1}}, \tag{4.4}$$

cf. (1.2), the same procedure would yield the theory of Jackiw and Polychronakos.

5 Discussion

While the Lagrangian (1.1) is first-order in the time derivative and the Hamiltonian (3.9) contains ugly square roots, ours are quadratic, as in ordinary relativistic scalar field theory. Now we show that, despite their rather different appearance, the two systems (1.1) and (3.4) can be transformed into each other. First, eliminating the auxiliary variable ρ using (3.6) for our energy (3.8) or (4.4) for the energy (3.9), respectively, both expressions become, for $a^2 = 2\lambda$,

$$\mathcal{H}^{\mathrm{JP}} = \mathcal{H} = a\frac{1 + (\vec{\nabla}\Theta)^2}{\sqrt{1 + (\vec{\nabla}\Theta)^2 - (\partial_\tau\Theta)^2}}. \tag{5.1}$$

Inspection of the other conserved quantities in (3.8) and (3.9) shows furthermore that they are actually the same when the fields are redefined as

$$\rho^{\mathrm{JP}} = \rho\,\partial_\tau\Theta, \qquad \Theta^{\mathrm{JP}} = -\Theta. \tag{5.2}$$

The equations of motion (1.2) and (3.5) go also into each other under (5.2). Similarly, the term in the Jackiw-Polychronakos Lagrangian (1.1) which is of the first-order in the time derivative is equivalent to $-\rho^{\mathrm{JP}}\partial_\tau\Theta^{\mathrm{JP}}$. Under (5.2) this becomes quadratic, namely $\rho\partial_\tau\Theta\partial^\tau\Theta$, so that the Lagrange densities (1.1) and (3.1) equivalent. The possibility of having differently-looking but secretely equivalent systems corresponds to the freedom of chosing the kinetic term [2].

Is the Jackiw-Polychronakos system equivalent to ours or not ? From the d-brane point of view, the answer is yes : the difference is merely the choice of the auxiliary variable. Viewed as fluid mechanical models, however, the densities are themselves observable; the two systems describe therefore different physics. We can start in fact with any solution Θ of the Born-Infeld system (1.3); then ρ and ρ^{JP} are given by (3.6) and (4.4), respectively. For $d \neq 1$, for example, choosing [2]

$$\Theta = -c\sqrt{c^2\tau^2 + \frac{x^2}{d-1}} \tag{5.3}$$

where we have restored the speed of light c, we get

$$\begin{aligned} \rho &= \sqrt{\frac{2\lambda}{d}}\frac{(d-1)}{|x|}\sqrt{\tau^2 + \frac{x^2}{c^2(d-1)}}, \\ \rho^{\mathrm{JP}} &= \sqrt{\frac{a^2}{d}}\frac{(d-1)}{|x|}\frac{\tau^2}{\sqrt{\tau^2 + \dfrac{x^2}{c^2(d-1)}}}, \end{aligned} \tag{5.4}$$

which are similar but still different. Letting $c \to \infty$, we recover, for both theories, the same non-relativistic solution, *viz.*

$$\rho = \sqrt{\frac{2\lambda}{d}}(d-1)\,\frac{|\tau|}{|x|}. \tag{5.5}$$

The $c \to \infty$ limit of the Jackiw-Polychronakos model is the non-relativistic Chaplygin gas [2]. This can also be seen in our framework : deforming the space-like reduction into lightlike amounts to taking the non-relativistic limit [7].

Let us mention, in conclusion, that our formalism can also be used to study the conformal properties of gas dynamics [10]. For the adiabatic potential $V(\varrho) \propto \rho^n$, the action (3.1) is readily seen to be invariant w. r.

t. the $(d+1,1)$ dimensional conformal group $\mathrm{O}(d+1,2)$ if and only if the polytropic exponent is

$$n = 1 + \frac{2}{d}. \tag{5.6}$$

(This can also be seen from the trace condition $T^{\mu}_{\ \mu} = 0$ of the energy-momentum tensor (3.7)).

In the free case, $\mathrm{O}(d+1,2)$ is a [field-dependent] symmetry also for the reduced system [5]. For $V \neq 0$, however, the potential is only consistent with equivariance,

$$\begin{aligned} \partial_\sigma \varrho = 0 \quad &\Longrightarrow \quad \varrho = \rho(x,\tau), \\ \partial_\sigma \theta = 1 \quad &\Longrightarrow \quad \theta = \Theta(x,\tau) + \sigma, \end{aligned} \tag{5.7}$$

rather than with the generalized condition (3.3). Equivariance reduces, however, the $(d+1,1)$ dimensional conformal symmetry to its mere [(d,1)-Poincaré]$\times\mathbf{R}$ subgroup, the $\mathbf{R}$ representing the vertical translations, whose associated conserved quantity is the "number" N. Let us recall that in the non-relativistic case the corresponding subgroup is the $(d,1)$ dimensional Schrödinger group [11, 5, 10].

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